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CONSTRUCTION OF SOLUTIONS FOR THE GENERALIZED HÉNON–HEILES SYSTEM WITH THE HELP OF THE PAINLEVÉ TEST

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Abstract

The Hénon–Heiles system in the general form has been considered. In a nonintegrable case with the help of the Painlevé test new solutions have been found as formal Laurent or Puiseux series, depending on three parameters. One of parameters determines a location of the singularity point, other parameters determine coefficients of series. It has been proved, that if absolute values of these two parameters are less or equal to unit, then obtained series converge in some ring. For some values of these parameters the obtained Laurent series coincide with the Laurent series of the known exact solutions.

Keywords: nonintegrable systems, the singularity analysis, a polynomial potential, the Laurent series, the Puiseux series, elliptic functions.

Вернов С.Ю.

ПОСТРОЕНИЕ РЕШЕНИЙ ОБОБЩЁННОЙ СИСТЕМЫ ХЕНОНА–ХЕЙЛЕСА С ПОМОЩЬЮ ТЕСТА ПЕНЛЕВЕ

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Аннотация

Рассматривается система Хенона–Хейлеса в общем виде. В неинтегрируемом случае с помощью теста Пенлеве найдены решения в виде формальных рядов Лорана или Пюизё, зависящих от трёх параметров. Один из параметров определяет положение особой точки, а два других — коэффициенты рядов. Доказано, что если эти два параметра по модулю меньше или равны единице, то ряды сходятся в некотором кольце. При определённых значениях этих параметров получаются ряды Лорана известных точных решений.

1 THE PAINLEVÉ PROPERTY AND INTEGRABILITY

A Hamiltonian system in a $2s$–dimensional phase space is called *completely integrable* (Liouville integrable) if it possesses s independent integrals which commute with respect to the associated Poisson bracket. When this is the case, the equations of motion are (in principal, at least) separable and solutions can be obtained by the method of quadratures.

When we study some mechanical or field theory problem, we imply that time and space coordinates are real, whereas the integrability of motion equations is connected with the behavior of their solutions as functions of complex time and (in the case of the field theory) complex spatial coordinates.

S.V. Kovalevskaya was the first, who proposed [1] to consider time as a complex variable and to demand that solutions of the motion equations have to be single-valued, meromorphic functions on the whole complex (time) plane. This idea gave a remarkable result: S.V. Kovalevskaya discovered a new integrable case (nowadays known as the Kovalevskaya's case) for the motion of a heavy rigid body about a fixed point [1] (see also [2]). The work of S.V. Kovalevskaya has shown the importance of application of the analytical theory of differential equations to physical problems. The essential stage of development of this theory was a classification of ordinary differential equations (ODE's) in order of types of singularities of their solutions. This classification has been made by P. Painlevé.

Let us formulate the Painlevé property for ODE's. Solutions of a system of ODE's are regarded as analytic functions, may be with isolated singular points [3, 4]. A singular point of a solution is said *critical* (as opposed to *noncritical*) if the solution is multivalued (single-valued) in its neighborhood and *movable* if its location depends on initial conditions[1].

Definition. A system of ODE's has ***the Painlevé property*** *if its general solution has no movable critical singular point* [5, 6].

An arbitrary solution of such system is single-valued in the neighborhood

[1]Solutions of a system with a time-independed Hamiltonian can have only movable singularities.

of its singular point t_0 and can be expressed as a Laurent series with a finite number of terms with negative powers of $t-t_0$. If a system has not the Painlevé property, but, after some change of variables, the obtained system possesses this property, then the initial system is said to have *the weak Painlevé property.*

Investigations of many dynamical systems, Hamiltonian [7–9] or dissipative (for example, the Lorenz systems [9–12]), show, that a system is completely integrable only for such values of parameters, at which it has the Painlevé property (or the weak Painlevé property). Arguments, which clarify the connection between the Painlevé analysis and the existence of motion integrals, are presented in [13, 14]. If the system misses the Painlevé property (has complex or irrational "resonances"), then the system cannot be "algebraically integrable-[15] (see also [16] and references therein). At the same time the integrability of an arbitrary system with the Painlevé property has yet to be proved. There is not an algorithm for construction of the additional integral by the Painlevé analysis. It is easy to give an example of an integrable system without the Painlevé property [17]: $H=\frac{1}{2}p^2+f(x)$, where $f(x)$ is a polynomial which power is not lower than five. The given system is trivially integrable, but its general solution is not a meromorphic function.

The Painlevé test is any algorithm designed to determine necessary conditions for a differential equation to have the Painlevé property. The original algorithm, developed by P. Painlevé and used by him to find all the second order ODE's with Painlevé property [6], is known as the α-method. The method of S.V. Kovalevskaya is not as general as the α–method, but much more simple[2].

In 1980, motivated by the work of S.V. Kovalevskaya [1], M.J. Ablowitz, A. Ramani and H. Segur [19] developed a new algorithm of the Painlevé test for ODE's. The remarkable property of this test is that it can be checked in a finite number of steps. They also were the first to point out the connection between the nonlinear partial differential equations (PDE's), which are soluble by the inverse scattering transform method, and ODE's with the Painlevé property. Subsequently the Painlevé property for PDE was defined and the corresponding Painlevé test (the WTC procedure) was constructed [20, 21] (see also [18, 22–25]). With the help of this test it has been found, that all PDE's, which are solvable by the inverse scattering transforms, have the Painlevé property, may be, after some change of variables. For many integrable PDE's, for example, the Korteweg–de-Vries equation [9], the Bäcklund transformations and the Lax

[2] Different variants of the Painlevé test are compared in [18, R. Conte paper]

representations result from the WTC procedure [21, 26]. Also, special solutions for certain nonintegrable PDE's were constructed using this algorithm [27, 28].

The algorithm for finding special solutions for ODE system in the form of a finite expansion in powers of unknown function $\varphi(t-t_0)$ has been constructed in [29]. The function $\varphi(t-t_0)$ and coefficients have to satisfy some system of ODE, often more simple than an initial one. This method has been used [30] to construct exact solutions for certain nonintegrable systems of ODE's.

The aim of this paper is to find new special solutions for the generalized Hénon–Heiles system using the Painlevé test. In distinction to [30] we obtain solutions as formal Laurent or Puiseux series and find domains of their convergence.

2 THE HÉNON–HEILES HAMILTONIAN

Let us consider the motion of a star in an axial-symmetric and time-independent potential. The motion equations admit two well-known integrals (energy and angular momentum) and would be solved by the method of quadratures if the third integral of motion is known. Due to the symmetry of the potential the considered system is equivalent to two-dimensional one. However, for many polynomial potentials the obtained system has not the second integral as a polynomial function.

In the 1960s, asymptotic methods [31, 32] have been developed to show either existence or absence of the third integral for some polynomial potentials. To answer the question about the existence of the third integral Hénon and Heiles [33] considered the behavior of numerically integrated trajectories. Emphasizing that their choice of potential does not proceed from experimental data, Hénon and Heiles have proposed the following Hamiltonian:

$$H=\frac{1}{2}(x_t^2+y_t^2+x^2+y^2)+x^2y-\frac{1}{3}y^3, \qquad (1)$$

because: on the one hand, it is analytically simple; this makes the numerical computations of trajectories easy; on the other hand, it is sufficiently complicated to give trajectories which are far from trivial. Indeed, for low energies the Hénon–Heiles system appears to be integrable, in so much as trajectories (numerically integrated) always lay on well-defined two-dimensional surfaces. On the other hand, for high energies many of these integral surfaces are destroyed, it points on absence of the third integral.

Subsequent numerical investigations [34, 35] show, that in the complex t-plane singular points of solutions of the motion equations group in self-similar spirals. It turns out extremely complicated distributions of singularities, forming a boundary, across which the solutions can not be analytically continued.

The generalized Hénon–Heiles system is described by the Hamiltonian:

$$H = \frac{1}{2}(x_t^2 + y_t^2 + \lambda x^2 + y^2) + x^2 y - \frac{C}{3}\, y^3 \qquad (1')$$

and the corresponding system of the motion equations:

$$\left\{ \begin{array}{lcl} x_{tt} & = & -\lambda x - 2xy, \\ y_{tt} & = & -y - x^2 + Cy^2, \end{array} \right. \qquad (2)$$

where $x_{tt} \equiv \frac{d^2x}{dt^2}$ and $y_{tt} \equiv \frac{d^2y}{dt^2}$, λ and C are numerical parameters.

Due to the Painlevé analysis the following integrable cases of (2) have been found:

$$\begin{array}{lll} \text{(i)} & C = -1, & \lambda = 1, \\ \text{(ii)} & C = -6, & \lambda \text{ is an arbitrary number}, \\ \text{(iii)} & C = -16, & \lambda = \frac{1}{16}. \end{array}$$

In contradiction to the case (i) the cases (ii) and (iii) are nontrivial, so the integrability of these cases had to be proved additionally. In the 1980's the required second integrals were constructed [36–39]. For integrable cases of the Hénon–Heiles system the Bäcklund transformations [29] and the Lax representations [22, 23, 40] have been found. In [41] the connection between the three integrable cases of the Hénon–Heiles system and some integrable partial differential equations was shown.

The Hénon–Heiles system is a model widely used in physics, in particular, in gravitation [42–44] and plasma theory [45]. The models, described by the Hamiltonian (1′) with some additional nonpolynomial terms, are actively studied [46–48] as well.

3 NONINTEGRABLE CASES

The general solutions of the Hénon–Heiles system are known only in integrable cases [48], in other cases search of new (exact or asymptotic) solutions is an actual problem.

The procedure for transformation the Hamiltonian to a normal form and for construction the second independent integral in the form of formal power series in the phase variables x, x_t, y and y_t (Gustavson integral) has been realized for the Hénon–Heiles system both in the original ($\lambda = 1$, $C = 1$) [32] (see also [49]) and in the general forms [50, 51]. Using the Bruno algorithm [52, 53] V.F. Edneral has constructed the Poincaré–Dulac normal form and found [54, 55] (provided that all phase variables are small) local families of periodic solutions. Recently it has been found that a local series around the singularities in the complex (time) plane can be transformed to some local series around the singularities at the fixed points in phase space and analyzed via normal forms theory [56, 57].

The Hénon–Heiles system as a system of two second order ODE's is equivalent to the fourth order equation[3]:

$$
\begin{aligned}
y_{tttt} &= (2C-8)y_{tt}y - (4\lambda+1)y_{tt} + 2(C+1)y_t^2 + \\
&+ \frac{20C}{3}y^3 + (4C\lambda - 6)y^2 - \lambda y - 4H,
\end{aligned} \tag{3}
$$

where H is the energy of the system.

To find a special solution of the given equation one can assume that y satisfies some more simple equation. For example, the well-known solutions in terms of the Weierstrass elliptic functions [58, 59] satisfy the following first-order differential equation:

$$
y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^2 + \tilde{\mathcal{C}}y + \tilde{\mathcal{D}}, \tag{4}
$$

where

$$
\tilde{\mathcal{A}} = \frac{2}{3}C, \qquad \tilde{\mathcal{B}} = -1, \qquad \tilde{\mathcal{C}} = 0 \quad \text{and} \quad \tilde{\mathcal{D}} = 2H \tag{4a}
$$

or

[3] For given $y(t)$ the function $x^2(t)$ is a solution of a linear equation. System (2) is invariant to exchange x to $-x$.

$$
\begin{aligned}
\tilde{\mathcal{A}} &= -\frac{4}{3},\\
\tilde{\mathcal{B}} &= \frac{1-(C+2)\lambda}{C+1},\\
\tilde{\mathcal{C}} &= -\frac{3C^2\lambda^2-3C^2\lambda+8C\lambda^2-7C\lambda-C+4\lambda^2-2\lambda-2}{3C^3+10C^2+11C+4},\\
\tilde{\mathcal{D}} &= \frac{24C^4H+104C^3H-9C^3\lambda^3+6C^3\lambda^2+3C^3\lambda}{4(3C^5+22C^4+60C^3+78C^2+49C+12)}+\\
&+\frac{168C^2H-30C^2\lambda^3+13C^2\lambda^2+16C^2\lambda+C^2}{4(3C^5+22C^4+60C^3+78C^2+49C+12)}+\\
&+\frac{120CH-28C\lambda^3+24C\lambda+4C+32H-8\lambda^3-4\lambda^2+8\lambda+4}{4(3C^5+22C^4+60C^3+78C^2+49C+12)}.
\end{aligned}
\tag{4b}
$$

$\tilde{\mathcal{D}}$ is proportional to energy H (arbitrary parameter), therefore, solutions $(4a)$ and $(4b)$ are two-parameter ones.

E.I. Timoshkova [60] generalized equation (4):

$$
y_t^2=\tilde{\mathcal{A}}y^3+\tilde{\mathcal{B}}y^2+\tilde{\mathcal{C}}y+\tilde{\mathcal{D}}+\tilde{\mathcal{G}}y^{5/2}+\tilde{\mathcal{E}}y^{3/2} \tag{5}
$$

and found new one-parameter sets of solutions of the Hénon–Heiles system in nonintegrable cases ($C=-\frac{4}{3}$ or $C=-\frac{16}{5}$, λ is an arbitrary number). These solutions (i.e. solutions with $\tilde{\mathcal{G}}\neq 0$ or $\tilde{\mathcal{E}}\neq 0$) are derived only at $\tilde{\mathcal{D}}=0$, therefore, substitution $y=\varrho^2$ gives:

$$
\varrho_t^2=\frac{1}{4}(\tilde{\mathcal{A}}\varrho^4+\tilde{\mathcal{G}}\varrho^3+\tilde{\mathcal{B}}\varrho^2+\tilde{\mathcal{E}}\varrho+\tilde{\mathcal{C}}). \tag{6}
$$

The general solution of (6) has one arbitrary parameter and can be expressed in elliptic functions.

In this paper I analyze system (2) at $C=-\frac{16}{5}$ and $\lambda=\frac{1}{9}$ (*the Solution 2.2* of the paper [60]). In this case equation (5) is:

$$
y_t^2+\frac{32}{15}y^3+\frac{4}{9}y^2\pm\frac{8i}{\sqrt{135}}y^{5/2}=0 \tag{7}
$$

and, depending on a choice of a sign before the last term, we obtain either (in case of sign $+$):

$$y=-\frac{5}{3\left(1-3\sin\left(\frac{t-t_0}{3}\right)\right)^2}\qquad\text{and}\qquad x^2=\frac{25\left(1-\sin\left(\frac{t-t_0}{3}\right)\right)}{9\left(1-3\sin\left(\frac{t-t_0}{3}\right)\right)^3};\tag{8.1}$$

or (in case of sign $-$):

$$y=-\frac{5}{3\left(1+3\sin\left(\frac{t-t_0}{3}\right)\right)^2}\qquad\text{and}\qquad x^2=\frac{25\left(1+\sin\left(\frac{t-t_0}{3}\right)\right)}{9\left(1+3\sin\left(\frac{t-t_0}{3}\right)\right)^3}.\tag{8.2}$$

4 RESULTS OF THE PAINLEVÉ TEST FOR THE HÉNON–HEILES SYSTEM

The Ablowitz–Ramani–Segur algorithm of the Painlevé test appears very useful to find asymptotic solutions as a formal Laurent series.

We assume that the behavior of solutions in a sufficiently small neighborhood of the singularity is algebraic, it means that x and y tend to infinity as some powers of $t-t_0$:

$$x=a_\alpha(t-t_0)^\alpha\qquad\text{and}\qquad y=b_\beta(t-t_0)^\beta,\tag{9}$$

where α, β, a_α and b_β are some constants. We assume that real parts of α and β are less then zero, and, of course, $a_\alpha\neq 0$ and $b_\beta\neq 0$.

If α and β are integer numbers, then substituting

$$x=a_\alpha(t-t_0)^\alpha+\sum_{j=1}^{N_{max}}a_{j+\alpha}(t-t_0)^{j+\alpha},\tag{10.1}$$

$$y=b_\beta(t-t_0)^\beta+\sum_{j=1}^{N_{max}}b_{j+\beta}(t-t_0)^{j+\beta}\tag{10.2}$$

one can transform the ODE system into a set of linear algebraic systems in coefficients a_k and b_k. In the general case one can obtain the exact solutions (in the form of formal Laurent series) only if one solves infinity number of systems ($N_{max}=\infty$). On the other hand, if one solves a finite number of systems one obtains asymptotic solutions. With the help of some computer algebra system, for example, the system **REDUCE** [61, 62], these systems can be solved step by step and asymptotic solutions can be automatically found with any accuracy. But previously one has to determine values of constants α, β, a_α and b_β and to analyze systems with zero determinants. Such systems correspond to new arbitrary constants or have no solutions. Powers at which new arbitrary constants enter are called *resonances*. The Painlevé test gives all information about possible dominant behaviors and resonances (see, for example, [9]). Moreover, the results of the Painlevé analysis point to cases, in which it is useful to include into expansion terms with fractional powers of $t - t_0$.

For the generalized Hénon-Heiles system there exist two possible dominant behaviors and resonance structures [9, 35]:

Case 1:	*Case 2*: $(\beta < \Re e(\alpha) < 0)$
$\alpha = -2$, $\beta = -2$, $a_\alpha = \pm 3\sqrt{2+C}$, $b_\beta = -3$, $r = -1,\ 6,\ \frac{5}{2} \pm \frac{\sqrt{1-24(1+C)}}{2}$.	$\alpha = \frac{1\pm\sqrt{1-48/C}}{2}$, $\beta = -2$, $a_\alpha = c_1$ **(an arbitrary number)**, $b_\beta = \frac{6}{C}$, $r = -1,\ 0,\ 6,\ \mp\sqrt{1-48/C}$.

In the Table the values of r denote resonances: $r = -1$ corresponds to arbitrary parameter t_0; $r = 0$ (in the *Case 2*) corresponds to arbitrary parameter c_1. Other values of r determine powers of t, to be exact, $t^{\alpha+r}$ for x and $t^{\beta+r}$ for y, at which new arbitrary parameters enter (as solutions of systems with zero determinants).

For integrability of system (2) all values of α and r have to be integer (or rational) and all systems with zero determinants have to have solutions at all values of included in them free parameters. It is possible only in the cases (i) — (iii).

At $C = -2$ (in the *Case 1*) $a_\alpha = 0$. It is the consequence of the fact that, contrary to our assumption, the behaviour of the solution in the neighborhood of a singular point is not algebraic, because its dominant term includes logarithm.

Those values of C, at which α and r are integer (or rational) numbers either only in the *Case 1* or only in the *Case 2*, are of interest for search of special solutions.

5 NEW SOLUTIONS

5.1 Finding of solutions in the form of formal Laurent series

Let us consider the Hénon–Heiles system with $C = -\frac{16}{5}$. In the *Case 1* some values of r are not rational, so it is a nonintegrable system. To find special asymptotic solutions let us consider the *Case 2*. In this case $\alpha = -\frac{3}{2}$ and $r = -1,\ 0,\ 4,\ 6$, hence, in the neighborhood of the singular point t_0 we have to seek x in such form that x^2 can be expand into Laurent series, beginning from $(t-t_0)^{-3}$. Let $t_0 = 0$, substituting

$$x = \sqrt{t}\left(c_1 t^{-2} + \sum_{j=-1}^{\infty} a_j t^j\right) \qquad \text{and} \qquad y = -\frac{15}{8}t^{-2} + \sum_{j=-1}^{\infty} b_j t^j$$

in (2), we obtain the following sequence of linear system in a_k and b_k:

$$\left\{\begin{array}{l} (k^2-4)a_k + 2c_1 b_k = -\lambda a_{k-2} - 2\displaystyle\sum_{j=-1}^{k-1} a_j b_{k-j-2} \\ ((k-1)k-12)b_k = -\, b_{k-2} - \displaystyle\sum_{j=-2}^{k-1} a_j a_{k-j-3} - \frac{16}{5}\sum_{j=-1}^{k-1} b_j b_{k-j-2}. \end{array}\right. \tag{11}$$

If $k = 2$ or $k = 4$, then the determinant of (11) is equal to zero. To determine a_2 and b_2 we have the following system:

$$\left\{\begin{array}{l} c_1(557056c_1^8 + (15552000\lambda - 4860000)c_1^4 + 864000000b_2 + \\ +\, 108000000\lambda^2 - 67500000\lambda + 10546875) = 0, \\ 818176c_1^8 + (15660000\lambda - 4893750)c_1^4 - \\ -\, 810000000b_2 - 6328125 = 0. \end{array}\right. \tag{12}$$

As one can see this system does not include terms, which are proportional to a_2, hence, a_2 is an arbitrary parameter (a constant of integration).

We discard the solution with $c_1 = 0$ and obtain the system in $\tilde{c}_1 \equiv c_1^4$ and b_2 with the following solutions:

$$\begin{aligned} \tilde{c}_1 &= \frac{1125(4\sqrt{35(2048\lambda^2-1280\lambda+387)}-1680\lambda+525)}{167552}, \\ b_2 &= -\frac{(10944\lambda-3420)\sqrt{35(2048\lambda^2-1280\lambda+387)}-4403456\lambda^2+2752160\lambda-789065}{117956608} \end{aligned}$$

or

$$\tilde{c}_1 = \frac{1125(-4\sqrt{35(2048\lambda^2-1280\lambda+387)}-1680\lambda+525)}{167552},$$

$$b_2 = \frac{(10944\lambda-3420)\sqrt{35(2048\lambda^2-1280\lambda+387)}-4403456\lambda^2+2752160\lambda-789065}{117956608}.$$

We obtain new constant of integration a_2, but we must fix c_1, so number of constants of integration is equal to 2. It is easy to verify that b_4 is an arbitrary parameter, because the corresponding system is equivalent to one linear equation. So, using Painlevé test, we obtain an asymptotic solution which depends on three parameters, namely t_0, a_2 and b_4.

Now asymptotic solutions can be obtained with arbitrary accuracy. For given λ one has to choose c_1 as one of the roots of system (12). After this the coefficients a_j and b_j can be found automatically with the help of some computer algebra system.

For example, if $\lambda = \frac{1}{9}$, then (12) has the following solutions:

$$\left\{\tilde{c}_1 = \frac{625}{128}, \quad b_2 = -\frac{1819}{663552}\right\}, \qquad \left\{\tilde{c}_1 = -\frac{8125}{23936}, \quad b_2 = -\frac{8700683}{1364926464}\right\}.$$

Taking into account, that system (2) is invariant to change x to $-x$, we obtain four types of formal solutions:

$$x = \sqrt{t}\left\{\frac{5\sqrt[4]{2}}{4}t^{-2} + \frac{25}{96\sqrt[4]{2}}t^{-1} - \frac{5\sqrt[4]{2}}{16} + \frac{5275}{663552\sqrt[4]{2}}t + a_2 t^2 \ldots\right\},$$

$$y = -\frac{15}{8}t^{-2} + \frac{5\sqrt{2}}{32}t^{-1} - \frac{205}{2304} + \frac{115\sqrt{2}}{13824}t - \frac{1819}{663552}t^2 + \tag{13.1}$$

$$+\left(\frac{741719\sqrt{2}}{1528823808} + \frac{5\sqrt[4]{2}}{12}a_2\right)t^3 + b_4 t^4 + \ldots;$$

$$x = \sqrt{t}\left\{\frac{5i\sqrt[4]{2}}{4}t^{-2} - \frac{25i}{96\sqrt[4]{2}}t^{-1} - \frac{5i\sqrt[4]{2}}{9216} - \frac{5275i}{663552\sqrt[4]{2}}t^1 + a_2 t^2 + \ldots\right\},$$

$$y = -\frac{15}{8}t^{-2} - \frac{5\sqrt{2}}{32}t^{-1} - \frac{205}{2304} - \frac{115\sqrt{2}}{13824}t - \frac{1819}{663552}t^2 - \tag{13.2}$$

$$-\left(\frac{741719\sqrt{2}}{1528823808} + \frac{5i\sqrt[4]{2}}{12}a_2\right)t^3 + b_4 t^4 + \ldots;$$

$$x=\sqrt{t}\left\{\frac{5\sqrt{2}}{4}\sqrt[4]{-\frac{13}{374}}t^{-2}+\frac{25i\sqrt{2431}}{17952}\sqrt[4]{-\frac{13}{374}}t^{-1}-\right.$$
$$\left.-\frac{38645\sqrt{2}}{574464}\sqrt[4]{-\frac{13}{374}}-\frac{7028575i\sqrt{2431}}{23203749888}\sqrt[4]{-\frac{13}{374}}t+a_2t^2+\ldots\right\},$$
$$y=-\frac{15}{8}t^{-2}+\frac{5i\sqrt{4862}}{5984}t^{-1}-\frac{69335}{430848}-\frac{37745i\sqrt{4862}}{483411456}t-\frac{8700683}{1364926464}t^2-$$
$$-\left(\frac{1148020763i\sqrt{13}\sqrt{374}}{3332429743915008}-\frac{5\sqrt{2}}{12}a_2\sqrt[4]{-\frac{13}{374}}\right)t^3+b_4t^4+\ldots;$$

$$x=\sqrt{t}\left\{\frac{5i\sqrt{2}}{4}\sqrt[4]{-\frac{13}{374}}t^{-2}+\frac{25\sqrt{2431}}{17952}\sqrt[4]{-\frac{13}{374}}t^{-1}-\right.$$
$$\left.-\frac{38645i\sqrt{2}}{574464}\sqrt[4]{-\frac{13}{374}}-\frac{7028575\sqrt{2431}}{23203749888}\sqrt[4]{-\frac{13}{374}}t+a_2t^2+\ldots\right\},$$
$$y=-\frac{15}{8}t^{-2}-\frac{5i\sqrt{4862}}{5984}t^{-1}-\frac{69335}{430848}+\frac{37745i\sqrt{4862}}{483411456}t-\frac{8700683}{1364926464}t^2-$$
$$-\left(\frac{1148020763\sqrt{13}\sqrt{374}}{3332429743915008}+\frac{5i\sqrt{2}}{12}a_2\sqrt[4]{-\frac{13}{374}}\right)t^3+b_4t^4+\ldots.$$

It is easy to verify that if

$$a_2=-\frac{21497\sqrt[4]{2}}{42467328}\qquad\text{and}\qquad b_4=-\frac{858455}{12039487488},$$

then series (13.1) are the Laurent series of (8.1). Also if

$$a_2=-\frac{21497i\sqrt[4]{2}}{42467328}\qquad\text{and}\qquad b_4=-\frac{858455}{12039487488},$$

then series (13.2) are the Laurent series of (8.2).

5.2 Convergence of the obtained series

When an asymptotic series is obtained the question about its convergence arises. It is known that a domain of Laurent series convergence is a ring. Let us find conditions, at which the obtained series converge in the following ring: $0<|t|\leqslant 1-\varepsilon$, where ε is any positive number.

The sum of a geometrical progression $S = \sum_{n=0}^{\infty} t^n = \dfrac{1}{1-t}$ is finite if $|t| \leqslant 1-\varepsilon$, hence, our series converge in the above-mentioned ring, if $\exists N$ such that $\forall n > N \quad |a_n| \leqslant 1$ and $|b_n| \leqslant 1$.

Let $|a_n| \leqslant 1$ and $|b_n| \leqslant 1$ for all $-1 \leqslant n < k$, then from (11), we obtain:

$$|a_k| \leqslant \frac{2k+2+|\lambda|+2|c_1|}{|k^2-4|}, \qquad |b_k| \leqslant \frac{21(k+2)}{5(k^2-k-12)}. \tag{14}$$

It is easy to see that there exists such N, that, if $|a_n| \leqslant 1$ and $|b_n| \leqslant 1$ for $-1 \leqslant n \leqslant N$, then $|a_n| \leqslant 1$ and $|b_n| \leqslant 1$ for $-1 \leqslant n < \infty$. N is a maximum from 8 and $1+\sqrt{|\lambda|+2|c_1|+7}$.

For example, if $\lambda = \frac{1}{9}$, then for any possible value of c_1, we obtain $N = 8$. It is easy to verify that if $|a_2| \leqslant 1$ and $|b_4| \leqslant 1$, then $|a_n| \leqslant 1$ and $|b_n| \leqslant 1$ for $-1 \leqslant n \leqslant 8$, and, hence, for an arbitrary n. Thus our Laurent series converge in the ring $0 < |t| \leqslant 1-\varepsilon$. Numerical analysis shows [63] that these series can also converge at absolute values of parameters more than unit. For other values of λ the convergence can be considered analogously.

6 GENERALIZATION OF SOLUTIONS IN TERMS OF THE WEIERSTRASS ELLIPTIC FUNCTIONS

Let us consider solutions (4). The values of parameters ($4a$) correspond to $x(t) \equiv 0$. Solutions ($4b$) correspond to the *Case 1* (see Table) and $x(t)$ can be expressed in terms of the Weierstrass elliptic functions. For some values of C these two-parameter solutions can be generalized. For example, if $C = -\frac{9}{8}$, then some resonaces are half-integer. Substituting

$$x = \sum_{k=-4}^{\infty} \tilde{a}_k t^{k/2} \qquad \text{and} \qquad y = \sum_{k=-4}^{\infty} \tilde{b}_k t^{k/2}$$

in (2), we obtain that $\tilde{a}_n$ and $\tilde{b}_n$ have to satisfy the following system:

$$\left\{ \begin{aligned} &\frac{n(n-2)-24}{4}\tilde{a}_n + 2\tilde{a}_{-4}\tilde{b}_n = -\lambda\tilde{a}_{n-4} - 2\sum_{k=-1}^{n-1} \tilde{a}_k\tilde{b}_{n-k-4}, \\ &2\tilde{a}_{-4}\tilde{a}_n + \frac{n(n-2)-27}{4}\tilde{b}_n = -\,\tilde{b}_{n-4} - \sum_{k=-3}^{n-1} \tilde{a}_k\tilde{a}_{n-k-4} - \frac{9}{8}\sum_{k=-3}^{n-1} \tilde{b}_k\tilde{b}_{n-k-4}, \end{aligned} \right.$$

where $\tilde{a}_{-4} = \pm\frac{3\sqrt{7}}{2\sqrt{2}}$ and $\tilde{b}_{-4} = -3$.

At any λ we obtain three-parameter solutions as formal Puiseux series. For example, if $\lambda = 1$ (and $\tilde{a}_{-4} = \frac{3\sqrt{7}}{2\sqrt{2}}$) the solution is the following ($t_0 = 0$):

$$\begin{aligned} x &= \tfrac{3\sqrt{7}}{2\sqrt{2}}t^{-2} + \tfrac{7}{8\sqrt{2}} + \tfrac{4}{\sqrt{14}}D_1 t^{3/2} + \tfrac{\sqrt{7}}{160\sqrt{2}}t^2 - \tfrac{15\sqrt{7}}{224\sqrt{2}}D_1 t^{7/2} - \\ &- \tfrac{\sqrt{7}}{2\sqrt{2}}D_2 t^4 - \tfrac{467\sqrt{7}}{8624\sqrt{2}}D_1^2 t^5 + \tfrac{1157\sqrt{7}}{430080\sqrt{2}}D_1 t^{11/2} + \tfrac{\sqrt{7}}{115200\sqrt{2}}t^6 + \ldots, \\ y &= -3t^{-2} - \tfrac{1}{4} + D_1 t^{3/2} - \tfrac{1}{80}t^2 - \tfrac{15}{128}D_1 t^{7/2} + \\ &+ D_2 t^4 - \tfrac{79}{616}D_1^2 t^5 + \tfrac{1157}{245760}D_1 t^{11/2} \tfrac{1}{57600}t^6 + \ldots. \end{aligned} \tag{15}$$

Using numerical calculations it is easy to show that if $|D_1| < 1$ and $|D_2| < 1$ then $|\tilde{a}_k| < 1$ and $|\tilde{b}_k| < 1$, for $-3 \leqslant k \leqslant 50$, except only $\tilde{a}_3 = \sqrt{\frac{8}{7}}D_1$. It is sufficient to prove that $|\tilde{a}_k| < 1$ and $|\tilde{b}_k| < 1$ for all $k > 50$ and, hence, our series converge in the ring $0 < |t| \leqslant 1 - \varepsilon$. If $D_1 = 0$ then y satisfies (4) with

$$\tilde{\mathcal{A}} = -\frac{4}{3}, \qquad \tilde{\mathcal{B}} = -1, \qquad \tilde{\mathcal{C}} = 0 \quad \text{and} \quad \tilde{\mathcal{D}} = \frac{16}{15}H, \tag{4b$'$}$$

and solution can be presented in terms of the Weierstrass elliptic functions.

7 CONCLUSION

Using the Painlevé analysis one can not only find integrable cases of dynamical systems, but also construct special solutions in nonintegrable cases.

We have found the special solutions of the Hénon–Heiles system with $C = -\frac{16}{5}$ as formal Laurent series, depending on three parameters. For some values of two parameters the obtained solutions coincide with the known exact solutions. At $C = -\frac{9}{8}$ two-parameter solutions in terms of the Weierstrass elliptic functions have been generalized to three-parameter ones. New solutions found as formal Puiseux series. For some values of λ the analysis of convergence of the obtained series has been made and it has been proved, that they have nonzero domain of convergence. Similar analysis can been made for any value of λ.

With the help of the Painlevé test particular asymptotic solutions as Laurent or Puiseux series can be found for the Hénon–Heiles system with some other values of C and λ. Just at these values of parameters the probability of finding of new exact solutions similar to the solutions found in [60] is great.

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