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GEOMETRICAL SPLITTING AND REDUCTION OF FEYNMAN DIAGRAMS

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Abstract

A geometrical approach to the calculation of N-point Feynman diagrams is reviewed. It is shown that the geometrical splitting yields useful connections between Feynman integrals with different momenta and masses. It is demonstrated how these results can be used to reduce the number of variables in the occurring functions.

Андрей Иванович Давыдычев

ГЕОМЕТРИЧЕСКОЕ РАЗБИЕНИЕ И УПРОЩЕНИЕ ДИАГРАММ ФЕЙНМАНА

Препринт НИИЯФ МГУ № 2016-2/890

Аннотация

В работе описан геометрический подход к вычислению N-точечных диаграмм Фейнмана. Показано, что геометрическое разбиение даёт полезные соотношения между диаграммами Фейнмана с разными импульсами и массами. Продемонстрировано, как эти результаты могут быть использованы для уменьшения числа переменных в получаемых функциях.

Geometrical splitting and reduction of Feynman diagrams

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Abstract. A geometrical approach to the calculation of N-point Feynman diagrams is reviewed. It is shown that the geometrical splitting yields useful connections between Feynman integrals with different momenta and masses. It is demonstrated how these results can be used to reduce the number of variables in the occurring functions.

1. Introduction

A geometrical interpretation of kinematic invariants and other quantities related to N-point Feynman diagrams (shown in figure 1) helps us to understand the analytical structure of the results for these diagrams. As an example, singularities of the general three-point function can be described pictorially through a tetrahedron constructed out of the external momenta and internal masses. Such a geometrical visualization can be used to derive Landau equations defining the positions of possible singularities [1] (see also in [2]).

In general the one-loop N-point diagrams (as shown in figure 1) depend on $\frac{1}{2}N(N-1)$ momentum invariants $k_{jl}^2 = (p_j - p_l)^2$ and N masses of the internal particles m_i. Here and below we follow the notation used in [3]; in particular, the powers of the internal scalar propagators are denoted as ν_i, and the space-time dimension is denoted as n, so that we can also deal with the dimensionally-regulated integrals with $n = 4 - 2\varepsilon$ [4]. Below we will mainly consider the cases when all $\nu_i = 1$.

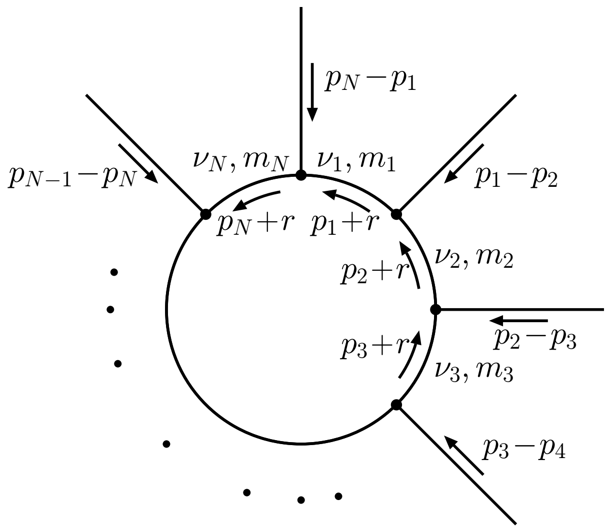

Figure 1. N-point diagram

In [5, 6, 7] it was demonstrated how such geometrical ideas could be used for an analytical calculation of one-loop N-point diagrams. For the geometrical interpretation, a "basic simplex" in N-dimensional Euclidean space is employed (a triangle for $N = 2$, a tetrahedron for $N = 3$, etc.), and the obtained results can be expressed in terms of an integral over a $(N-1)$-dimensional spherical (or hyperbolic) simplex, which corresponds to the intersection of the basic simplex and the unit hypersphere (or the corresponding hyperbolic hypersurface), with a weight function depending on the angular distance θ between the integration point and the point 0, corresponding to the height of the basic simplex (see in [5]). For $n = N$ this weight function is equal to 1, and the results simplify: for the case $n = N = 3$ see in [8], and for the case $n = N = 4$ see in [9, 10]. Other interesting examples of using the geometrical approach can be found, e.g., in [11].

In this paper we will show that the natural way of splitting the basic simplex, as prescribed within the geometrical approach discussed above, leads to a reduction of the effective number of independent variables in separate contributions obtained as a result of such splitting.

2. Two-point function

For the two-point function, there is only one external momentum invariant k_{12}^2, and the sides of the corresponding basic triangle are m_1, m_2 and $K_{12} \equiv \sqrt{k_{12}^2}$, as shown in figure 2a. The angle τ_{12} between the sides m_1 and m_2 is defined through $\cos\tau_{12} \equiv c_{12} = (m_1^2 + m_2^2 - k_{12}^2)/(2m_1m_2)$, and (in the spherical case) the integration goes over the arc τ_{12} of the unit circle, as shown in figure 2b.

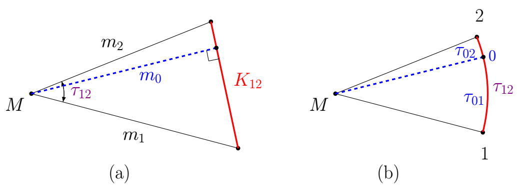

Figure 2. Two-point case: (a) the basic triangle and (b) the arc τ_{12}.

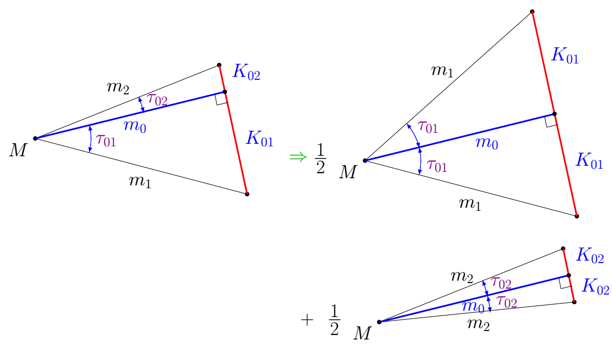

Figure 3. Two-point function: reduction to equal-mass integrals.

For splitting we use the height of the basic triangle, m_0, and obtain two triangles with the sides $(m_1, m_0, K_{01} \equiv \sqrt{k_{01}^2})$ and $(m_2, m_0, K_{02} \equiv \sqrt{k_{02}^2})$, respectively. Here $m_0 = m_1 m_2 \sin\tau_{12}/\sqrt{k_{12}^2}$, $k_{01}^2 = (k_{12}^2 + m_1^2 - m_2^2)^2/(4k_{12}^2)$ and $k_{02}^2 = (k_{12}^2 - m_1^2 + m_2^2)^2/(4k_{12}^2)$ (note that $k_{01}^2 = m_1^2 - m_0^2$ and $k_{02}^2 = m_2^2 - m_0^2$). Each of the resulting integrals can be associated with a two-point function, and we arrive at the following decomposition:

$$J^{(2)}\left(n; 1, 1 | k_{12}^2; m_1, m_2\right) = \frac{1}{2k_{12}^2}\Big\{(k_{12}^2 + m_1^2 - m_2^2) J^{(2)}\left(n; 1, 1 | k_{01}^2; m_1, m_0\right) + (k_{12}^2 - m_1^2 + m_2^2) J^{(2)}\left(n; 1, 1 | k_{02}^2; m_2, m_0\right)\Big\} . \quad (1)$$

This is an example of a functional relation between integrals with different momenta and masses, similar to those described in [12]. Moreover, using the geometrical relation shown in figure 3, we can represent the right-hand side in terms of the equal-mass integrals:

$$J^{(2)}\left(n; 1, 1 | k_{12}^2; m_1, m_2\right) = \frac{1}{4k_{12}^2}\Big\{(k_{12}^2 + m_1^2 - m_2^2) J^{(2)}\left(n; 1, 1 | 4k_{01}^2; m_1, m_1\right) + (k_{12}^2 - m_1^2 + m_2^2) J^{(2)}\left(n; 1, 1 | 4k_{02}^2; m_2, m_2\right)\Big\} . \quad (2)$$

Let us look at the number of variables. In the original integral $J^{(2)}\left(n; 1, 1 | k_{12}^2; m_1, m_2\right)$ we have three independent variables: two masses and one momentum invariant (out of them we can construct two dimensionless variables). In the integral $J^{(2)}\left(n;1,1|k_{01}^2;m_1,m_0\right)$ we have one extra condition on the variables, $k_{01}^2=m_1^2-m_0^2$, so that we get two independent variables (i.e., one dimensionless variable). The same is valid for $J^{(2)}\left(n;1,1|4k_{01}^2;m_1,m_1\right)$, where the extra condition is due to two equal masses. Therefore, the result for the two-point function in arbitrary dimension can be expressed in terms of a combination of functions of a single dimensionless variable: indeed, we know that it can be presented in terms of the Gauss hypergeometric function ${}_2F_1$ (see, e.g., in [5, 13]) whose ε-expansion is known to any order [14, 15].

3. Three-point function

For the three-point function, there are three external momentum invariants, k_{12}^2, k_{13}^2 and k_{23}^2, and the sides of the corresponding basic tetrahedron are m_1, m_2, m_3, $K_{12}\equiv\sqrt{k_{12}^2}$, $K_{13}\equiv\sqrt{k_{13}^2}$ and $K_{23}\equiv\sqrt{k_{23}^2}$, as shown in figure 4a. The angles τ_{12}, τ_{13} and τ_{23} between the sides m_1, m_2 and m_3 are defined through $\cos\tau_{jl}\equiv c_{jl}=(m_j^2+m_l^2-k_{jl}^2)/(2m_jm_l)$, and (in the spherical case) the integration extends over the spherical triangle 123 of the unit sphere, see in figure 4b.

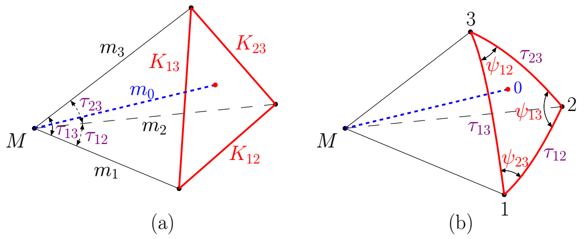

Figure 4. Three-point case: (a) the basic tetrahedron and (b) the solid angle.

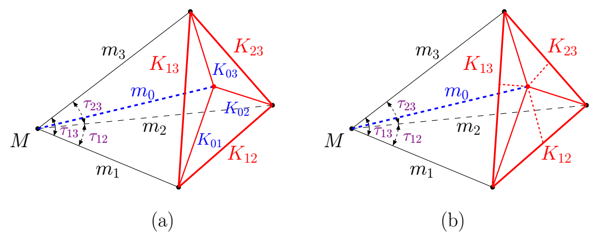

Figure 5. (a) Splitting the basic tetrahedron into three tetrahedra and (b) further splitting into six tetrahedra.

For the splitting we use the height of the basic tetrahedron, m_0, and obtain three tetrahedra, as shown in figure 5a. One of them has the sides m_1, m_2, m_0, $K_{12}\equiv\sqrt{k_{12}^2}$, $K_{01}\equiv\sqrt{k_{01}^2}$ and $K_{02}\equiv\sqrt{k_{02}^2}$, and the sides for the others can be obtained by permutation of the indices. Here $k_{01}^2=m_1^2-m_0^2$, $k_{02}^2=m_2^2-m_0^2$, $k_{03}^2=m_3^2-m_0^2$, and $m_0=m_1m_2m_3\sqrt{D^{(3)}/\Lambda^{(3)}}$, where $\Lambda^{(3)}=\frac{1}{4}\left[2k_{12}^2k_{13}^2+2k_{13}^2k_{23}^2+2k_{23}^2k_{12}^2-(k_{12}^2)^2-(k_{13}^2)^2-(k_{23}^2)^2\right]$, and $D^{(3)}=\det\|c_{jl}\|$ is the Gram determinant, see in [5, 6] for more details. Each of the resulting integrals can be associated with a specific three-point function, and we arrive at the following decomposition:

$$
\begin{aligned}
J^{(3)}\left(n;1,1,1|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right) = \frac{m_1^2m_2^2m_3^2}{\Lambda^{(3)}}\Bigg\{&\frac{F_1^{(3)}}{m_1^2}J^{(3)}\left(n;1,1,1|k_{23}^2,k_{03}^2,k_{02}^2;m_0,m_2,m_3\right)\\
&+\frac{F_2^{(3)}}{m_2^2}J^{(3)}\left(n;1,1,1|k_{03}^2,k_{13}^2,k_{01}^2;m_1,m_0,m_3\right)\\
&+\frac{F_3^{(3)}}{m_3^2}J^{(3)}\left(n;1,1,1|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)\Bigg\}, \qquad (3)
\end{aligned}
$$

with

$$
F_3^{(3)}=\frac{1}{4m_1^2m_2^2}\left[k_{12}^2\left(k_{13}^2+k_{23}^2-k_{12}^2+m_1^2+m_2^2-2m_3^2\right)-(m_1^2-m_2^2)\left(k_{13}^2-k_{23}^2\right)\right], \qquad (4)
$$

etc., so that $\sum_{i=1}^{3}(F_i^{(3)}/m_i^2)=\Lambda^{(3)}/(m_1^2m_2^2m_3^2)$.

By dropping perpendiculars onto the sides $K_{12} \equiv \sqrt{k_{12}^2}$, etc., each of the resulting tetrahedra can be split into two, so that in total we get six "birectangular" tetrahedra, as shown in figure 5b. Furthermore, for each of them we can use the geometrical relation similar to one shown in figure 3, reducing them to the integrals with two equal masses:

$$
\begin{aligned}
&J^{(3)}\left(n;1,1,1|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right) \\
&\quad = \frac{1}{2k_{12}^2}\Bigg\{(k_{12}^2+m_1^2-m_2^2)J^{(3)}\left(n;1,1,1|k_{01}^2,k_{01}^2,\frac{(k_{12}^2+m_1^2-m_2^2)^2}{k_{12}^2};m_1,m_1,m_0\right) \\
&\qquad +(k_{12}^2-m_1^2+m_2^2)J^{(3)}\left(n;1,1,1|k_{02}^2,k_{02}^2,\frac{(k_{12}^2-m_1^2+m_2^2)^2}{k_{12}^2};m_2,m_2,m_0\right)\Bigg\}. \qquad (5)
\end{aligned}
$$

Let us analyze the number of variables. In the integral $J^{(3)}\left(n;1,1,1|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$ we have six independent variables: three masses and three momentum invariants (out of them we can construct five dimensionless variables). In $J^{(3)}\left(n;1,1,1|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$ we have two extra conditions on the variables, $k_{01}^2 = m_1^2 - m_0^2$ and $k_{02}^2 = m_2^2 - m_0^2$, so that we get four independent variables (i.e., three dimensionless variables). For the integral $J^{(3)}\left(n;1,1,1|k_{01}^2,k_{01}^2,(k_{12}^2+m_1^2-m_2^2)^2/k_{12}^2;m_1,m_1,m_0\right)$ we have three relations, with an additional condition due to the two equal masses. Therefore, the result for the three-point function in arbitrary dimension should be expressible in terms of a combination of functions of two dimensionless variables: indeed, we know that it can be presented in terms of the Appell hypergeometric function F_1 (see, e.g., in [6, 16, 17]).

4. Four-point function

For the four-point function, there are six external momentum invariants. Out of them, k_{12}^2, k_{23}^2, k_{34}^2 and k_{14}^2 are the squared momenta of the external legs, whilst k_{13}^2 and k_{24}^2 correspond to the Mandelstam variables s and t. The sides of the corresponding basic four-dimensional simplex are m_1, m_2, m_3, m_4, and six additional sides $K_{jl} \equiv \sqrt{k_{jl}^2}$, as shown in figure 6a. The six angles τ_{jl} between the corresponding sides m_j and m_l are defined through $\cos\tau_{jl} \equiv c_{jl} = (m_j^2+m_l^2-k_{jl}^2)/(2m_jm_l)$, and (in the spherical case) the integration extends over the spherical tetrahedron 1234 of the unit hypersphere, as shown in figure 6b (for the hyperbolic case one can use analytic continuation).

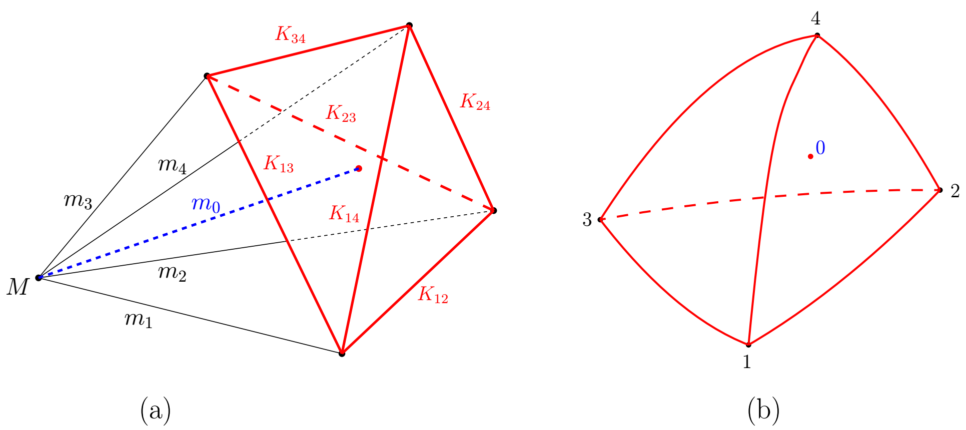

Figure 6. Four-point case: (a) the basic simplex and (b) the spherical tetrahedron.

For splitting we use the height of the basic simplex, m_0, and obtain four simplices, as shown in figure 7a. One of them has the sides m_1, m_2, m_3, m_0, $K_{12} \equiv \sqrt{k_{12}^2}$, $K_{13} \equiv \sqrt{k_{13}^2}$,

$K_{23} \equiv \sqrt{k_{23}^2}$, $K_{01} \equiv \sqrt{k_{01}^2}$, $K_{02} \equiv \sqrt{k_{02}^2}$ and $K_{03} \equiv \sqrt{k_{03}^2}$, and the sides of the others can be obtained by permutation of the indices. As before, $k_{0i}^2 = m_i^2 - m_0^2$ $(i = 1,2,3,4)$, whereas $m_0 = m_1 m_2 m_3 m_4 \sqrt{D^{(4)}/\Lambda^{(4)}}$, where $D^{(4)} = \det \|c_{jl}\|$ and $\Lambda^{(4)} = \det \|(k_{j4} \cdot k_{l4})\|$, see in [5] for more details. Each of the four resulting integrals can be associated with a certain four-point function. At the next step, in each of the four tetrahedra (drawn in red) we drop the perpendiculars onto the triangle sides, as shown in figure 7b, splitting each of them into three, and then dividing each of the resulting tetrahedra into two, by dropping perpendiculars onto the $\sqrt{k_{jl}^2}$ sides, as shown in figure 7c. As a result of this splitting, we get 24 simplices. The corresponding steps of splitting the spherical tetrahedron are shown in figure 8.

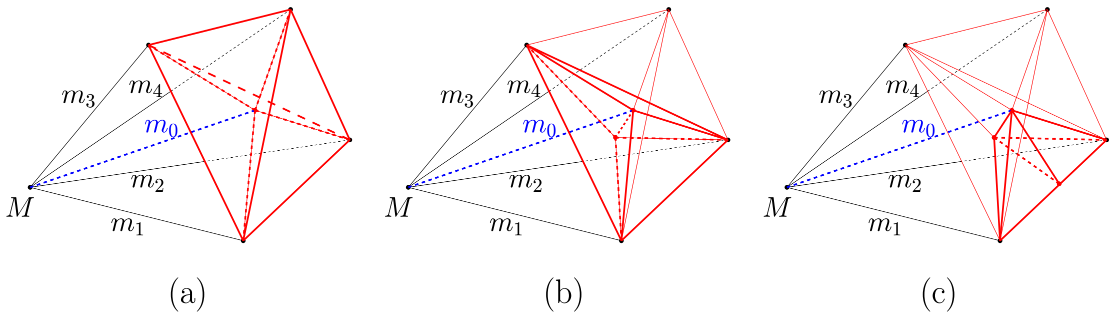

Figure 7. Four-point case: splitting the basic four-dimensional simplex.

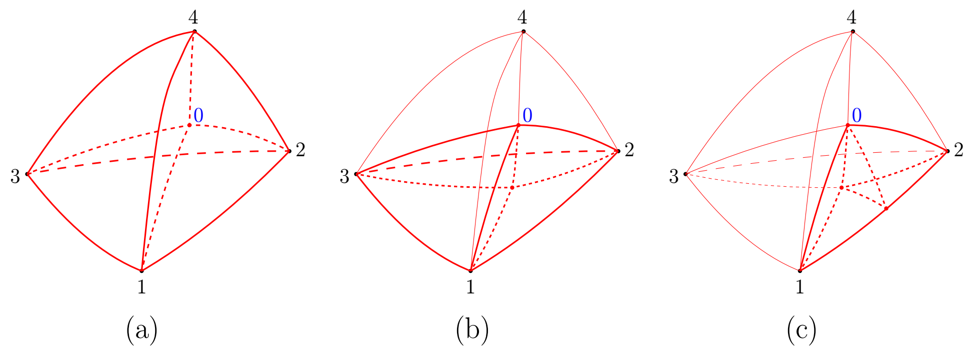

Figure 8. Four-point case: splitting the spherical tetrahedron.

Let us look at the number of variables. In the integral $J^{(4)}\left(n;1,1,1,1|\{k_{jl}^2\};\{m_i\}\right)$ we have ten independent variables: four masses and six momentum invariants (out of them we can construct nine dimensionless variables). After the first step (figure 7a) we have three extra conditions on the variables, $k_{01}^2 = m_1^2 - m_0^2$, $k_{02}^2 = m_2^2 - m_0^2$ and $k_{03}^2 = m_3^2 - m_0^2$, so that we get seven independent variables (i.e., six dimensionless variables). After the second step (figure 7b), we get two extra conditions due to the right triangles, and after the third step (figure 7c) we get one more condition. As a result, for each of the 24 resulting four-point functions we have six relations, so that we end up with four independent variables (i.e., three dimensionless variables). Therefore, the result for the four-point function in arbitrary dimension should be expressible in terms of a combination of functions of three dimensionless variables, such as, e.g., Lauricella functions and their generalizations (see, e.g., in [17, 18]).

5. General remarks and conclusions

Using a geometrical approach, we can relate the one-loop N-point Feynman diagrams to certain volume integrals in non-Euclidean geometry. Geometrical splitting provides a straightforward way of reducing general integrals to those with lesser number of independent variables. In this way, we can predict the set and the number of these variables in the resulting integrals. Furthermore, it allows us to derive functional relations between integrals with different momenta and masses.

Numbers of dimensionless variables in separate contributions for N-point diagrams, before and after the splitting, are summarized in the table.

Table 1. Number of variables before and after the splitting

	total # of dimensionless variables	# of splitting pieces	reduced # of variables
$N=2$	$3-1=2$	2	1
$N=3$	$6-1=5$	6	2
$N=4$	$10-1=9$	24	3
arbitrary N	$\frac{1}{2}(N-1)(N+2)$	$N!$	$N-1$ (?)

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ГЕОМЕТРИЧЕСКОЕ РАЗБИЕНИЕ И УПРОЩЕНИЕ ДИАГРАММ ФЕЙНМАНА

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